NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH AND APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFICATION: Bachelor of Science in Applied Mathematics and Statistics	
QUALIFICATION CODE: 07BAMS	**LEVEL:** 7
COURSE CODE: RAN701S	**COURSE NAME:** REAL ANALYSIS
SESSION: JULY 2019	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

SUPPLEMENTARY/SECOND OPPORTUNITY EXAMINATION QUESTION PAPER	
EXAMINER	PROF. G. HEIMBECK
MODERATOR:	PROF. F. MASSAMBA

INSTRUCTIONS

1. Answer ALL the questions in the booklet provided.
2. Show clearly all the steps used in the calculations.
3. All written work must be done in blue or black ink and sketches must be done in pencil.

PERMISSIBLE MATERIALS

1. Non-programmable calculator without a cover.

THIS QUESTION PAPER CONSISTS OF 4 PAGES (Including this front page)

Question 1 [14 marks]

a) When is a set of real numbers bounded? State the definition. [3]

b) Let $X \subset \mathbb{R}$. Prove that X is bounded if and only if there exists some $K > 0$ such that $|x| < K$ for all $x \in X$. [6]

c) Prove that the union of two bounded sets of real numbers is bounded. [5]

Question 2 [17 marks]

Consider the following sequence:

$$a_n := \frac{1 - \frac{1}{n}}{1 + \frac{1}{n}} \text{ for all } n \in \mathbb{N}.$$

In answering the following questions, you are supposed to give reasons.

a) Is the sequence $(a_n)_{\mathbb{N}}$ monotonic? [5]

b) Is the sequence

$$\left(\frac{2n-1}{2n+1}\right)_{\mathbb{N}}$$

a subsequence of $(a_n)_{\mathbb{N}}$? [7]

c) Is the sequence $(a_n)_{\mathbb{N}}$ convergent? [5]

Question 3 [14 marks]

a) State Bernoulli's inequality. [3]

b) Prove that

$$\lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right)^n = 1.$$

[5]

c) Show that the sequence $\left((1 - \frac{1}{n})^n\right)_{\mathbb{N}}$ is convergent and find its limit. [6]

Question 4 [19 marks]

a) Let $l, m \in \mathbb{N}$ such that $l \leq m$. If $a \in \mathbb{R} - \{1\}$, show that

$$\sum_{k=l}^{m} a^k = \frac{a^l - a^{m+1}}{1-a}.$$

What is the sum if $a = 1$? [7]

b) Let $b = (b_n)_{\mathbb{N}}$ be a sequence of real numbers. Prove, by induction on n, that

$$\sum_{k=1}^{n} b_{2k-1} + \sum_{k=1}^{n} b_{2k} = \sum_{k=1}^{2n} b_k$$

for all $n \in \mathbb{N}$. [5]

c) For which real numbers $q \in \mathbb{R}$ is $\sum q^k$ convergent? Prove your assertion. If $\sum q^k$ is convergent, find the sum $\sum_{k=1}^{\infty} q^k$. [7]

Question 5 [11 marks]

Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ be a function.

a) When is f continuous at $a \in \mathbb{R}$? State the definition. [3]

b) Assume that a is an isolated point of X.

i) Prove that f is continuous at a. [5]

ii) Is $\lim_{x \to a} f = f(a)$ true? Explain your answer. [3]

Question 6 [9 marks]

a) What is an interval of $\mathbb{R}$? State the definition. [3]

b) Show that $\mathbb{R} - \{0\}$ is not an interval of $\mathbb{R}$. [3]

c) State the Intermediate Value Theorem. [3]

Question 7 [15 marks]

Let $a, b \in \mathbb{R}$ such that $a < b$ and let $f, g\colon [a,b] \to \mathbb{R}$ be functions which are differentiable on (a,b) and continuous at a and b. Consider $h\colon [a,b] \to \mathbb{R}$ defined by

$$h(x) := (g(b) - g(a))f(x) - (f(b) - f(a))g(x).$$

a) Verify that h is differentiable on (a,b) and continuous at a and b. [4]

b) Show that $h(a) = h(b)$ and apply the theorem of Rolle. [6]

c) If f' and g' do not have a common zero on (a,b) and $g(a) \neq g(b)$, show that, there exists some $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

[5]

End of the question paper